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Group classification of the time fractional nonlinear Poisson equation

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Abstract. In this paper, we present a complete symmetry analysis for the time fractional nonlinear Poisson equation. A minimal symmetry algebra for any arbitrary function $f(u)$ is obtained. A group classification is then carried out by investigating for $f(u)$ that give larger symmetry algebras. Symmetries are obtained and some exact solutions are constructed.

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1. Introduction

Fractional differential equation models are well known in the fields of science and engineering. They are broadly used in applied mechanical and electrical engineering to analyse the properties of materials. In geology, it is applied to obtain rheological properties of rocks as well as a neuron model in biology [21, 20]. In recent years, many methods have been developed to analyse and find analytical solutions to fractional differential equations like Laplace transforms, Fourier transforms and the decomposition method [32, 31, 16, 18, 19, 17, 33]. A Poisson equation belongs to the class of well-known elliptic partial differential equations that have several applications in both science and engineering domains. Physically, the Poisson equation describes how a function diffuses in space and is general form of the Laplace differential equation. A function that satisfies the Laplace equation is called the harmonic function and is vital in the fields of fluid dynamics, astronomy, complex analysis and electromagnetism, among others [23, 15, 8].

A modern approach to the applications of Lie Symmetry theory remains a powerful tool to study both deterministic and stochastic differential equations [27]-[34]. Recently, Lie group theory has been extended to the class of fractional differential equations for the purposes of linearization, reduction in the number of independent variables and finding analytical solutions, [36, 5, 40, 7, 2, 35, 24, 13, 11, 12, 10]. It is worth noticing that the theory is still developing.

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The aim of this article is to use classical Lie symmetry theory to present a complete group classification of the time fractional nonlinear Poisson equation with the Riemann-Liouville derivative, i.e.,

$$\frac{\partial^\alpha u}{\partial t^\alpha} + \frac{\partial^2 u}{\partial x^2} = f(u), \;\; f_{uu} \neq 0, \tag{1}$$

where $\frac{\partial^\alpha u}{\partial t^\alpha} = u_\alpha$ is the fractional derivative and $u = u(t,x)$. To achieve this, we begin by finding the Lie symmetry algebra for an arbitrary function $f(u)$, and later look for all possible functions $f(u)$ for which larger symmetry algebras exist.

There is no unique definition of fractional derivatives [21, 20]. In this paper, we use a Riemann-Liouville version given by

$$D_t^\alpha u(t,x) = \begin{cases} \frac{\partial^n u}{\partial t^n}, & \alpha = n \in \mathbb{N} \\ \frac{1}{\Gamma(n-\alpha)} \frac{\partial^n}{\partial t^n} \int_0^t \frac{u(\mu,x)}{(t-\mu)^{\alpha+1-n}} d\mu, & n-1 < \alpha < n, n \in \mathbb{N}, \end{cases} \tag{2}$$

where Γ is a gamma function, and D_t^α satisfies the following properties [21, 20, 38]:

$$D^\alpha t^\varsigma = \frac{\Gamma(\varsigma+1)}{\Gamma(\varsigma+1-\alpha)} t^{\varsigma-\alpha}, \;\; \alpha > 0, \;\; t > 0,$$
$$D^\alpha 1 = \frac{t^{-\alpha}}{\Gamma(1-\alpha)}, \;\; \alpha \geq 0, \;\; t > 0,$$

and

$$D_t^\alpha g(h(t)) = (D_g^\alpha g(h))_{h=h(t)} (D_t^1 h(t))^\alpha.$$

The fractional Riemann-Liouville integral of order $\alpha > 0$ of a function [20, 6]

$$f(t) : (0,\infty) \to \mathbb{R}$$

is defined by

$$I^\alpha (f(t)) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-\tau)^{\alpha-1} f(\tau) d\tau.$$

The fractional Riemann-Liouville integral has the following properties [20, 6]:

(I) For $\alpha > 0, \;\; t > 0$,

$$D^\alpha (I^\alpha (f(t))) = f(t),$$

that is, the Riemann-Liouville fractional derivative is the left inverse of the Riemann-Liouville fractional integral of the same order.

(II) If the fractional derivative of a function of order α is integrable, then

$$D^\alpha (I^\alpha (f(t))) = f(t) - \sum_{n}^{j=1} \left[D^{\alpha-j} f(t)\right]_{t=0} \frac{t^{\alpha-j}}{\Gamma(\alpha-j+1)},$$

where $n = [\alpha] + 1$ and if $m < 0$, $D^m f(t)$ is defined as $D^m f(t) = I^{-m} (f(t))$.

2. Lie point symmetry analysis

We consider a one-parameter Lie group of infinitesimal transformations in (t, x, u) given by:

$$\begin{aligned} x^* &= x + \epsilon\xi(t, x, u) + O(\epsilon) \\ t^* &= t + \epsilon\tau(t, x, u) + O(\epsilon) \\ u^* &= u + \epsilon\phi(t, x, u) + O(\epsilon), \end{aligned} \tag{3}$$

where ϵ is the group parameter with the corresponding infinitesimal generator of the Lie algebra of the form

$$X = \xi(t, x, u)\frac{\partial}{\partial x} + \tau(t, x, u)\frac{\partial}{\partial t} + \phi(t, x, u)\frac{\partial}{\partial u}.$$

We prolong the infinitesimal generator X to X^α to define how u_α and u_{xx} are transformed [22, 11, 12], i.e.,

$$X^\alpha = X + \eta^{xx}\frac{\partial^2}{\partial x^2} + \eta^\alpha\frac{\partial}{\partial u_\alpha}, \tag{4}$$

where η^{xx} and η^α are the prolonged infinitesimals of order 2 and α, respectively, defined as:

$$\begin{aligned} \eta^{xx} =&\phi_{xx} + (2\phi_{ux} - \xi_{xx})u_x - \tau_{xx}u_t + (\phi_{uu} - 2\xi_{ux})u_x^2 - 2\tau_{ux}u_xu_t - \xi_{uu}u_x^3 \\ &- \tau_{uu}u_x^2u_t + (\phi_u - 2\xi_x)u_{xx} - 2\tau_xu_{xt} - 3\xi_uu_{xx}u_x - \tau_uu_{xx}u_t - 2\tau_uu_{xt}u_x, \end{aligned}$$

and

$$\eta^\alpha = D_t^\alpha(\phi) + \xi D_t^\alpha(u_x) - D_t^\alpha(\xi u_x) + D_t^\alpha(D_t(\tau)u) - D_t^{\alpha+1}(\tau u) + \tau D_t^{\alpha+1}(u).$$

The α-th prolonged infinitesimal can be rewritten as

$$\eta^\alpha = \frac{\partial^\alpha\phi}{\partial t^\alpha} + (\phi_u - \alpha D_t(\tau))\frac{\partial^\alpha u}{\partial t^\alpha} - u\frac{\partial^\alpha\phi_u}{\partial t^\alpha} + \beta,$$

where

$$\begin{aligned} \beta =&\sum_{n=1}^{+\infty}\left[\binom{\alpha}{n}\frac{\partial^n\phi_u}{\partial t^n} - \binom{\alpha}{n+1}D_t^{n+1}(\tau)\right]D_t^{\alpha-n}(u) - \sum_{n=1}^{+\infty}\binom{\alpha}{n}D_t^n(\xi)D_t^{\alpha-n}(u_x) \\ &+ \sum_{n=2}^{+\infty}\sum_{m=2}^{n}\sum_{k=2}^{m}\sum_{r=0}^{k-1}\binom{\alpha}{n}\binom{n}{m}\binom{k}{r}\frac{1}{k!} \\ &\times\frac{t^{n-\alpha}}{\Gamma(n+1-\alpha)}[-u]^r\frac{\partial^m}{\partial t^m}(u^{k-r})\frac{\partial^{n-m+k}\phi}{\partial t^{n-m}\partial u^k}. \end{aligned}$$

The one-parameter Lie group of transformation (3) is admitted by (1) if the invariance criterion

$$X^\alpha\left(\frac{\partial^\alpha u}{\partial t^\alpha} + \frac{\partial^\beta u}{\partial x^\beta} - f(u, v)\right)\Big|_{(3)} = 0 \tag{5}$$

is satisfied for every solution of (1). Evaluating equation (5), we have

$$\eta^{\alpha} + \eta^{xx} - \phi f_u \Big|_{u_\alpha = f(u) - u_{xx}} = 0. \tag{6}$$

By substituting η^{α}, η^{xx} in (6) and equating to zero, the coefficients of various derivatives of u, i.e., u_x, u_{xx}, u_t, u_{xt},... and $D_t^{\alpha-n}u$, $D_t^{\alpha-n}u_x$... for $n = 1, 2...$, we obtain a simplified system of determining equations as follows:

$$\xi_u = \tau_x = \tau_u = \phi_{uu} = \xi_t = 0, \tag{7}$$

$$2\phi_{ux} - \xi_{xx} = 0, \tag{8}$$

$$\binom{\alpha}{n} \frac{\partial^n \phi_u}{\partial t^n} - \binom{\alpha}{n+1} D_t^{n+1}(\tau) = 0, \quad \forall n \in \mathbb{N}, \tag{9}$$

$$2\xi_x - \alpha\tau_t = 0 \tag{10}$$

and

$$\frac{\partial^\alpha \phi}{\partial t^\alpha} + \phi_u f - \alpha f \tau_t - u \frac{\partial^\alpha \phi_u}{\partial t^\alpha} + \phi_{xx} - \phi f_u = 0. \tag{11}$$

Next, we solve the above system (7)–(11) to obtain the minimal symmetry algebra for any nonlinear function $f(u)$ as follows.

From equations (8) and (10)

$$\xi_{xx} = \phi_{ux} = 0, \tag{12}$$

using (7) and (12), we obtain the following spatial infinitesimal:

$$\xi = c_1 x + c_2. \tag{13}$$

Similarly, equations (10) and (13) lead to a temporal infinitesimal given by

$$\tau = \frac{2c_1}{\alpha} t + c_3. \tag{14}$$

From (7), (9) and (12), we have

$$\phi = ru + h(t, x). \tag{15}$$

Differentiating (11) with respect to u gives

$$\phi f_{uu} + \alpha \tau_t f_u = 0. \tag{16}$$

Differentiating (16) with respect to u, leads to

$$\phi_u f_{uu} + \phi f_{uuu} + \alpha \tau_t f_{uu} = 0 \tag{17}$$

Eliminating ϕ by using (16) and (17), we have

$$\phi_u f_{uu}^2 + \alpha \tau_t \left(f_{uu}^2 - f_u f_{uuu} \right) = 0. \tag{18}$$

Finally, differentiating (18) with respect to u we obtain

$$2\phi_u f_{uu} f_{uuu} + \alpha \tau_t \left(f_{uu} f_{uuu} - f_u f_{uuuu} \right) = 0, \tag{19}$$

and using (18) and (19) to eliminate ϕ_u, we have:

$$\alpha\left(f_{uu}^2 f_{uuu} - 2f_u f_{uuu}^2 + f_u f_{uu} f_{uuuu}\right)\tau_t = 0. \tag{20}$$

To avoid any restrictions on $f(u)$, we consider $\tau_t = 0$, and it then follows from (16), (13) and (14), respectively, that

$$\phi = 0, \quad \xi = c_2, \quad \tau = c_3 \tag{21}$$

for any arbitrary function $f(u)$. The lower limit of the integral in the definition of the fractional derivative (2) is fixed, it requires that the manifold $t = 0$ is invariant, i.e.,

$$\tau(t,x)\Big|_{t=0} = 0. \tag{22}$$

Hence, equation (21) reduces to

$$\phi = 0, \quad \xi = c_2, \quad \tau = 0.$$

Therefore, the fractional Poisson equation (1) admits a one-dimensional minimal symmetry algebra spanned by

$$X_1 = \frac{\partial}{\partial x}.$$

3. Group classification

To search for a non linear function(s) $f(u)$ that may give a larger symmetry algebra, we assume $\tau_t \neq 0$. Using (14), (15), (18) and (20), we have

$$(2c_1 + r)f_{uu}^2 - 2c_1 f_u f_{uuu} = 0 \tag{23}$$

and

$$f_{uu}^2 f_{uuu} - 2f_u f_{uuu}^2 + f_u f_{uu} f_{uuuu} = 0. \tag{24}$$

It can then be deduced from (15) and (16) that

$$\phi = ru + s, \tag{25}$$

where r, s are arbitrary constants. Solving (23) by using symbolic software *Maple* we have the following cases:

(i) if $r = 0$, we have an exponential function in the form:

$$f(u) = c_0 e^{ku} + c, \ k \neq 0. \tag{26}$$

(ii) if $r = -\frac{2c_1}{m} \neq 0$

$$f(u) = \begin{cases} (au+b)^{m+1} + c & \text{if } m \neq -1, 0 \\ \frac{\log(au+b)}{a} + c & \text{if } m = -1. \end{cases} \tag{27}$$

Similarly, to solve equation (24), i.e.,

$$f_{uu}^2 f_{uuu} - 2f_u f_{uuu}^2 + f_u f_{uu} f_{uuuu} = 0,$$

we start by substituting $H = f_u$ in (24) to have

$$HH_u H_{uuu} - 2HH_{uu}^2 + H_u^2 H_{uu} = 0. \tag{28}$$

From (28), we consider the following cases:

- If $H_{uu} = 0$, then equation (24) has a quadratic solution, that is,

$$f(u) = au^2 + bu + c.$$

- If $H_{uu} \neq 0$, dividing equation (28) by $HH_u H_{uu}$, gives

$$\frac{H_{uuu}}{H_{uu}} - 2\frac{H_{uu}}{H_u} + \frac{H_u}{H} = 0,$$

  which can be integrated to have

$$\log\left|\frac{H_{uu}H}{H_u^2}\right| = k. \tag{29}$$

  From (29), we deduce that

$$\frac{H_{uu}}{H_u} - \frac{dH_u}{H} = 0. \tag{30}$$

  Integrating (30) leads to $\ln|\frac{H_u}{H^d}| = c_4$, implying that $\frac{H_u}{H^d} = c_5$ and hence we have the following cases:

  - If $d = 1$, we get $f(u)$ of the form

$$f(u) = c_0 e^{ku} + c, \ k \neq 0.$$

  - If $d = 2$, then H is of the form $H = (au + b)^{-1}$, which implies

$$f(u) = \frac{\log|au + b|}{a} + c, \ a \neq 0.$$

  - If $d \neq 1, 2$, then H is of the form $H = (au + b)^{\frac{1}{1-c}}$ which gives

$$f(u) = (au + b)^n + c, \ n \neq 0, 1, 2.$$

Remark 1. *Clearly, these solutions are similar to the solutions to equation (23). Therefore, to look for extra symmetries, it is sufficient to use (14), (27), (26) and (25) in*

$$\phi_u f - \alpha f \tau_t - \phi f_u = 0. \tag{31}$$

3.1. $f(u) = c_0 e^{ku} + c,\ k \neq 0$

From (31), (26), (25), (14) and (22), we found out that a larger symmetry algebra is possible if $c = 0$ and $s = -\frac{2c_1}{k}$, i.e., for $f(u) = c_0 e^{-ku}$ the fractional Poisson equation (1) admits a two-dimensional algebra

$$X_1 = \frac{\partial}{\partial x}, \qquad X_2 = x\frac{\partial}{\partial x} + \frac{2t}{\alpha}\frac{\partial}{\partial t} - \frac{2}{k}\frac{\partial}{\partial u}$$

obtained from

$$\xi = c_1 x + c_2, \quad \tau = \frac{2c_1}{\alpha}t, \quad \phi = -\frac{2c_1}{k}.$$

The admitted symmetry algebra forms a Lie algebra, the Lie bracket relations are $[X_1, X_2] = X_1$ and $[X_2, X_1] = -X_1$.

3.2. $f(u) = (au + b)^{m+1} + c$ if $m \neq -1, 0$

Similarly, using (31), (27), (25), (14) and (22), we obtained a larger symmetry algebra if $c = 0$, $r = -\frac{2c_1}{m}$ and $s = -\frac{2c_1 b}{am}$, and the infinitesimals are given as follows:

$$\xi = c_1 x + c_2, \quad \tau = \frac{2c_1}{\alpha}t, \quad \phi = -\frac{2c_1}{am}(au + b).$$

That is, for $f(u) = (au + b)^{m+1}$, fractional Poisson equation (1) admits a two-dimensional algebra

$$X_1 = \frac{\partial}{\partial x}, \qquad X_2 = x\frac{\partial}{\partial x} + \frac{2t}{\alpha}\frac{\partial}{\partial t} - \frac{2}{am}(au + b)\frac{\partial}{\partial u}. \tag{32}$$

The Lie bracket relations of the infinitesimal generators (32) are given as

$$[X_1, X_2] = X_1, \ \textit{and}\ [X_2, X_1] = -X_1,$$

which clearly satisfy the Lie algebra properties.

3.3. $f(u) = \frac{\log|au+b|}{a} + c$, $a \neq 0$

From equation (31), using (27), (25) and (14), we note that no extra symmetry is possible.

The results of the group classification are summarized in Theorem 1 below.

Theorem 1. *The minimal symmetry algebra of the nonlinear space-time fractional Poisson equation (1)*

$$\frac{\partial^\alpha u}{\partial t^\alpha} + \frac{\partial^2 u}{\partial x^2} = f(u),\ f_{uu} \neq 0$$

is spanned by one-dimensional infinitesimal generators

$$X_1 = \frac{\partial}{\partial x}$$

for any arbitrary function $f(u)$. The larger symmetry algebra is possible with the functions in Table 1.

$f(u)$	Extra infinitesimal symmetry generators
$f(u) = c_0 e^{ku},\ k \neq 0$	$X_3 = x\frac{\partial}{\partial x} + \frac{2t}{\alpha}\frac{\partial}{\partial t} - \frac{2}{k}\frac{\partial}{\partial u}$
$f(u) = (au+b)^{m+1},\ m \neq -1, 0$	$X_3 = x\frac{\partial}{\partial x} + \frac{2t}{\alpha}\frac{\partial}{\partial t} - \frac{2}{am}(au+b)\frac{\partial}{\partial u}.$

Table 1: *Extra symmetry generators*

4. Reductions and exact solutions

In this section, we use the admitted Lie symmetries of the nonlinear space-time fractional Poisson equation to obtain symmetry reductions, thus constructing some exact solutions where possible.

Definition 1. *Given an infinitesimal generator*

$$X = \tau(t,x,u)\frac{\partial}{\partial t} + \xi(t,x,u)\frac{\partial}{\partial x} + \phi(t,x,u)\frac{\partial}{\partial u}, \tag{33}$$

the characteristic equation associated with the differential operator (33) is

$$\frac{dt}{\tau} = \frac{dx}{\xi} = \frac{du}{\phi}.$$

This can be integrated to obtain similarity variables that are invariant under the symmetries generated by X [4, 14].

4.1. $f(u) = (au+b)^{m+1},\ m \neq -1, 0$

Example 1. *Reduction and the exact solution to*

$$D_t^\alpha u + u_{xx} = au^{m+1}.$$

Using the sub-algebra X_1 we have the similarity variables as

$$z = t, \quad u = \psi(t),$$

where $\psi(t)$ is the solution to the reduced fractional ODE

$$D_t^\alpha \psi(t) = a\psi^{m+1}(t).$$

This can be solved to get [20, 22];

$$\psi(t) = \frac{1}{a}\left(\frac{\Gamma(\alpha+1)}{\Gamma(\frac{(m+1)\alpha}{m}+1)}\right)^{\frac{1}{m}} t^{-\frac{\alpha}{m}},\ m > -1,\ m \neq 0.$$

Example 2. *Consider the Poisson equation*

$$D_t^\alpha u + u_{xx} = (au)^{m+1}, \quad m \neq -1, 0. \tag{34}$$

The characteristic equation corresponding to subalgebra X_2 is

$$\alpha\frac{dt}{2t}=\frac{dx}{x}=-m\frac{du}{2u}.$$

Solving the above equation leads to the following invariant solution:

$$z=xt^{-\frac{\alpha}{2}},\ u=\psi(z)t^{-\frac{\alpha}{m}}. \tag{35}$$

Substituting transformation (35) into (34) leads to

$$\left(P_{\frac{2}{\alpha}}^{1-\alpha-\frac{\alpha}{m},\alpha}\psi\right)(z)+\psi_{zz}(z)=(a\psi(z))^{m+1},$$

where $\left(P_{\frac{2}{\alpha}}^{1-\alpha+\frac{\alpha}{m},\alpha}\psi\right)(z)$ is the well-known Erdelyi-Kober fractional differential operator [21, 20].

Proof. For $0<\alpha<1$, similarity transformation (39) can be written using the definition of Riemann-Liouville fractional derivatives as follows:

$$\frac{\partial^\alpha u}{\partial t^\alpha}=\frac{1}{\Gamma(1-\alpha)}\frac{\partial}{\partial t}\int_0^t(t-s)^{-\alpha}s^{-\frac{\alpha}{m}}\psi(xs^{-\frac{\alpha}{2}})ds. \tag{36}$$

Let $w=\frac{t}{s}$; then equation (36) can be transformed into

$$\begin{aligned}
\frac{\partial^\alpha u}{\partial t^\alpha}&=\frac{1}{\Gamma(1-\alpha)}\frac{\partial}{\partial t}\int_1^\infty\left(t-\frac{t}{w}\right)^{-\alpha}\left(\frac{t}{w}\right)^{-\frac{\alpha}{m}}\psi(zw^{\frac{\alpha}{2}})\frac{t}{w^2}dw\\
&=\frac{1}{\Gamma(1-\alpha)}\frac{\partial}{\partial t}\left[t^{-\alpha+1-\frac{\alpha}{m}}\int_1^\infty(w-1)^{-\alpha}w^{\frac{\alpha}{m}-2+\alpha}\psi(zw^{\frac{\alpha}{2}})dw\right]\\
&=\left[t^{-\alpha+1-\frac{\alpha}{m}}\left(K_{\frac{2}{\alpha}}^{1-\frac{\alpha}{m},1-\alpha}\psi\right)(z)\right]\\
&=t^{-\alpha-\frac{\alpha}{m}}\left(1-\frac{\alpha}{m}-\alpha-z\alpha\frac{d}{dz}\right)\left[\left(K_{\frac{2}{\alpha}}^{1-\frac{\alpha}{m},1-\alpha}\psi\right)(z)\right]\\
&=t^{-(\frac{\alpha}{m}+\alpha)}\left(P_{\frac{2}{\alpha}}^{1-\alpha-\frac{\alpha}{m},\alpha}\psi\right)(z),
\end{aligned}$$

where $\left(P_{\frac{2}{\alpha}}^{1-\alpha,\alpha}\psi\right)(z)$ is the well-known Erdelyi-Kober fractional differential operator defined in [21, 20]:

$$\left(P_{\frac{2}{\alpha}}^{1-\alpha,\alpha}\psi\right)(z)=\prod_{j=0}^{i-1}\left(\zeta+j-\frac{z}{\sigma}\frac{d}{dz}\right)\left(K_\sigma^{\zeta+\alpha,i-\alpha}\psi\right)(z),\qquad z,\psi,\alpha>0$$

$$j=\begin{cases}[\alpha]+1 & \text{if } \alpha\neq\mathbb{N}\\ \alpha & \text{if } \alpha=\mathbb{N},\end{cases}$$

with

$$\left(K_\sigma^{\zeta,\alpha}\psi\right)(z)=\begin{cases}\frac{1}{\Gamma(\alpha)}\int_0^\infty(w-1)^{\alpha-1}w^{-(\zeta+\alpha)}\psi(zw^{\frac{1}{\sigma}})dw & \text{if } \alpha>0\\ \psi(z) & \text{if } \alpha=0.\end{cases}$$

□

4.2. $f(u) = c_0 e^{ku}$

Example 3. *Consider equation (1) with $f(u) = c_0 e^{ku}$. The invariant solution corresponding to the infinitesimal generator X_1 is*

$$u = \psi(t),$$

which reduces (4) to a nonlinear fractional ODE

$$D_t^\alpha(\psi(t)) = c_0 e^{k\psi(t)}. \tag{37}$$

Taking the fractional integral on both sides of equation (37) and assuming that $0 < \alpha < 1$, we have

$$\psi(t) = \frac{c_0}{\Gamma(\alpha)} \int_0^t (t-\mu)^{\alpha-1} e^{k\psi(\mu)} d\mu. \tag{38}$$

Every solution of (37) is also a solution to the integral equation (38) [20, 6].

Example 4. *Solving a characteristic equation corresponding to X_2:*

$$\alpha \frac{dt}{2t} = \frac{dx}{x} = -k\frac{du}{2}.$$

Solving $\alpha\frac{dt}{2t} = \frac{dx}{x}$ and $\frac{dx}{x} = -k\frac{du}{2}$, we have similarity transformations

$$u = \frac{2(\psi(z) - \ln(x))}{k}, \quad z = xt^{-\frac{\alpha}{2}}, \tag{39}$$

where $\psi(x)$ is the solution to the reduced nonlinear fractional differential equation

$$\left(P_{\frac{2}{\alpha}}^{1-\alpha,\alpha}\psi\right)(z) + \psi_{zz}(z) + \frac{1}{z^2} = \frac{kc_0}{2z^2} e^{\psi(z)}.$$

The Erdelyi-Kober fractional differential operator $\left(P_{\frac{2}{\alpha}}^{1-\alpha,\alpha}\psi\right)(z)$ is defined in Example 2.

Remark 2. *Since for any arbitrary nonlinear function $f(u)$ the minimal symmetry algebra X_1 is admitted, we consider a function $f(u) = u^2 + k_4$, which can be transformed by using X_1 to*

$$D_t^\alpha \psi(t) = \psi^2(t) + k_4. \tag{40}$$

Fractional ODE (40) has the following solutions [20, 22, 40], where k_4 is an arbitrary constant:

$$u = \psi(t) = \begin{cases} \sqrt{k_4}\tan(\sqrt{k_4}t, \alpha) & \text{if } k_4 > 0 \\ -\sqrt{k_4}\cot(\sqrt{k_4}t, \alpha) & \text{if } k_4 > 0 \\ -\sqrt{-k_4}\tanh(-\sqrt{-k_4}t, \alpha) & \text{if } k_4 < 0 \\ -\sqrt{-k_4}\coth(-\sqrt{-k_4}t, \alpha) & \text{if } k_4 < 0 \\ \frac{\Gamma(\alpha+1)}{\Gamma(2\alpha+1)} t^{-\alpha} & \text{if } k_4 = 0. \end{cases} \tag{41}$$

5. Conclusion

We presented a complete group classification of a nonlinear time fractional Poisson equation via Lie group transformation techniques. For any arbitrary function $f(u)$, a two-dimensional algebra was obtained and we proved that a four-dimensional symmetry algebra is possible only for the cases of exponential and polynomial functions. Furthermore, using the Lie symmetry analysis we were able to transform the nonlinear space-time fractional Poisson equation for some exponential and polynomial functions to one independent variable equation and as a result, some exact solutions were obtained. In Table 2, we summarize invariant solutions, reduced equations and the corresponding exact solutions.

Representing equations	Generator	Invariants solutions	Reduced equations	Exact solution
$D_t^\alpha u + u_{xx} = c_0 e^{ku}$	X_1	$u = \psi(t)$	$D_t^\alpha(\psi(t)) = c_0 e^{k\psi}$	$u(t) = \psi(t)$.
$D_t^\alpha u + u_{xx} = c_0 e^{ku}$	X_2	$u = \frac{2(\psi(z)-\ln(x))}{k}$, $z = xt^{-\frac{\alpha}{2}}$	$\left(P_{\frac{2}{\alpha}}^{1-\alpha,\alpha}\psi\right)(z) + \psi_{zz}(z) + \frac{1}{z^2} = \frac{kc_0}{2z^2}e^{\psi(z)}$	$u = \psi(z)$
$D_t^\alpha u + u_{xx} = au^{m+1}$, $m > -1$, $m \neq 0$	X_1	$u = \psi(t)$	$D_t^\alpha \psi(t) = a\psi^{m+1}(t)$	$u(t) = \frac{1}{a}\left(\frac{\Gamma(\alpha+1)}{\Gamma(\frac{(m+1)\alpha}{m}+1)}\right)^{\frac{1}{m}} \times t^{-\frac{\alpha}{m}}$
$D_t^\alpha u + u_{xx} = (au)^{m+1}$, $m \neq -1, 0$	X_2	$z = xt^{-\frac{\alpha}{2}}$, $u = \psi(z)t^{-\frac{\alpha}{m}}$	$\left(P_{\frac{2}{\alpha}}^{1-\alpha-\frac{\alpha}{m},\alpha}\psi\right)(z) + \psi_{zz}(z) = (a\psi(z))^{m+1}$	$u = \psi(z)$

Table 2: *Table of solutions*

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